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Problem of screw dislocation in a non-homogeneous transversely isotropic annular disc

By Sanjib Kumar Chakrabarti

Department of Mathematics, Chandernagore College, West Bengal, India

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Expressions for stresses and displacements in the case of a screw dislocation in an annular disc of non-homogeneous transversely isotropic material have been calculated in this paper. The amount of energy necessary for forming the dislocation is also calculated. Stresses have been calculated numerically for different cases.

Introduction

Effects of dislocations on crystals were studied by Forty (1951), Dawson & Vand (1950, 1951). Thereafter Eshelby & Stroh (1951) discussed different cases of straight screw dislocation in a thin plate and disc and in an infinite body. In the present case, the problem of screw dislocation in a non- homogeneous transversely isotropic annular disc has been considered.

Solution of the Problem

Here we use the cylindrical co-ordinates r, θ, z. u_r, u_θ, u_z are the displacement components and τ_{rr}, $\tau_{r\theta}$, $\tau_{\theta z}$, etc., the stress components, c_{44}, c_{66} being the elastic constants Let the disc be bounded by $r_1 \leqq r \leqq r_2$ and $z = \pm d$.

Now we know that for a screw dislocation in an infinite transversely isotropic body along the Z-axis, the non-vanishing displacement and stress components are

$$u_z : \frac{b}{2\pi} \cdot \theta,$$

$$\tau_{\theta z} = \frac{c_{66} b}{2\pi} \qquad \ldots \quad (1)$$

For screw dislocation in a disc, the tractions on the plane sz $= \pm d$ must vanish. We have for transversely isotropic material (Love 1944)

$$\frac{\partial u_\theta}{\partial z}$$

$$\tau_{r\theta} = c_{66} \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \qquad \ldots \quad (2)$$

Let us suppose

$$c_{44} = c'_{44} r^l, \; c_{66} = c'_{66} r^l, \qquad \dots \quad (3)$$

c'_{44}, c'_{66}, l being constants.

The equation of equilibrium that does not vanish is

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0. \qquad \dots \quad (4)$$

This equation with (2) and (3) reduces to

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{l+1}{r} \frac{\partial u_\theta}{\partial r} - (l+1)\frac{u_\theta}{r^2} + K^2 \frac{\partial^2 u_\theta}{\partial z^2} = 0, \qquad \dots \quad (5)$$

where
$$K^2 = \frac{c'_{44}}{c'_{66}}.$$

Let $u_\theta = e^{imz} \,.\, U(r)$, m being an arbitrary constant.

Then (5) becomes

$$\frac{\partial^2 U}{\partial r^2} + \frac{l+1}{r} \frac{\partial U}{\partial r} - (l+1)\frac{U}{r^2} - K^2 m^2 U = 0.$$

Again, the substitution $U = r^p \,.\, V$ reduces the equation to

$$\frac{\partial^2 V}{\partial r^2} + \frac{l+1+2p}{r} \frac{\partial V}{\partial r} + \left\{ \frac{(p-1)(l+1+p)}{r^2} - K^2 m^2 \right\} V = 0, \quad \text{which for}$$

$$p = -\frac{l}{2}, \quad r = \frac{t}{km} \quad \text{becomes} \quad \frac{\partial^2 V}{\partial t^2} + \frac{1}{t} \frac{\partial V}{\partial t} - \left\{ \frac{\left(\frac{l}{2}+1\right)^2}{t^2} + 1 \right\} V = 0.$$

The solution of this equation is $V = AI_{\frac{l}{2}+1}(t) + BK_{\frac{l}{2}+1}(t)$.

Therefore,
$$u_\theta = e^{imz} r^{-\frac{l}{2}} \,.\, \left[AI_{\frac{l}{2}+1}(Kmr) + BK_{\frac{l}{2}+1}(Kmr) \right] \qquad \dots \quad (6)$$

Thus in our case the non-vanishing displacement and stress components can be written as

$$u_\theta = -\frac{b}{2\pi} \frac{z}{r} + r^{-\frac{l}{2}} \sum_{n \text{ odd}} \left[A_n I_{\frac{l}{2}+1}\left(\frac{Kn\pi r}{2d} \right) + B_n K_{\frac{l}{2}+1}\left(\frac{Kn\pi r}{2d} \right) \right] \sin \frac{n\pi z}{2d},$$

$$u_z = \frac{b}{2\pi} \theta,$$

$$\tau_{\theta z} = \frac{c'_{44}\pi}{2d} r^{\frac{l}{2}} \sum_{n\ odd} n\left[A_n I_{\frac{l}{2}+1}\left(\frac{Kn\pi r}{2d}\right) + B_n K_{\frac{l}{2}+1}\left(\frac{Kn\pi r}{2d}\right)\right] \cos\frac{n\pi z}{2d},$$

$$\tau_{r\theta} = \frac{c'_{66}b}{\pi} r^{l-z}\ z + \frac{c'_{66}K\pi}{2d} r^{\frac{l}{2}} \sum_{n\ odd} n\left[A_n I_{\frac{l}{2}+2}\left(\frac{Kn\pi r}{2d}\right)\right.$$

$$\left. - B_n K_{\frac{l}{2}+2}\left(\frac{Kn\pi r}{2d}\right)\right] \sin\frac{n\pi z}{2d} \quad \ldots (7)$$

The conditions on the boundary are $\tau_{r\theta} = 0$ on $r = r_1$ and on $r = r_2$. These give rise to

$$A_n I_{\frac{l}{2}+2}\left(\frac{Kn\pi r_1}{2d}\right) - B_n K_{\frac{l}{2}+2}\left(\frac{Kn\pi r_1}{2d}\right) = -\frac{16bd^2}{K\pi^4 n^3} r_1^{\frac{l}{2}-2} \sin\frac{n\pi}{2},$$

and $$A_n I_{\frac{l}{2}+2}\left(\frac{Kn\pi r_2}{2d}\right) - B_n K_{\frac{l}{2}+2}\left(\frac{Kn\pi r_2}{2d}\right) = -\frac{16bd^2}{K\pi^4 n^3} r_2^{\frac{l}{2}-2} \sin\frac{n\pi}{2}.$$

Solving, we get

$$A_n = \frac{16bd^2}{K\pi^4 n^3} \sin\frac{n\pi}{2}\,.$$

$$\frac{r_2^{\frac{l}{2}-2} K_{\frac{l}{2}+2}\left(\frac{Kn\pi r_1}{2d}\right) - r_1^{\frac{l}{2}-2} K_{\frac{l}{2}+2}\left(\frac{Kn\pi r_2}{2d}\right)}{I_{\frac{l}{2}+2}\left(\frac{Kn\pi r_1}{2d}\right) K_{\frac{l}{2}+2}\left(\frac{Kn\pi r_2}{2d}\right) - I_{\frac{l}{2}+2}\left(\frac{Kn\pi r_2}{2d}\right) K_{\frac{l}{2}+2}\left(\frac{Kn\pi r_1}{2d}\right)}, \quad \ldots (8)$$

$$B_n = \frac{16bd^2}{k\pi^4 n^3} \sin\frac{n\pi}{2}\,.$$

$$\frac{r_2^{\frac{l}{2}-2} I_{\frac{l}{2}+2}\left(\frac{Kn\pi r_1}{2d}\right) - r_1^{\frac{l}{2}-2} I_{\frac{l}{2}+2}\left(\frac{Kn\pi r_2}{2d}\right)}{I_{\frac{l}{2}+2}\left(\frac{Kn\pi r_1}{2d}\right) K_{\frac{l}{2}+2}\left(\frac{Kn\pi r_2}{2d}\right) - I_{\frac{l}{2}+2}\left(\frac{Kn\pi r_2}{2d}\right) K_{\frac{l}{2}+2}\left(\frac{Kn\pi r_1}{2d}\right)} \quad \ldots (8)$$

Thus with these values of A_n, B_n, the displacement and stresses are given by (7).

The energy required to form the dislocation in the annulus is given by

$$W = \tfrac{1}{2}\, b \int_{-d}^{+d} dz \int_{r_1}^{r_2} \tau_{\theta z}\, dr$$

$$= \sum_{n\ odd} c'_{44} b \sin \frac{n\pi}{2} \int_{r_1}^{r_2} r^{\frac{l}{2}} \left[A_n I_{\frac{l}{2}+1} \left(\frac{Kn\pi r}{2d} \right) + B_n K_{\frac{l}{2}+1} \left(\frac{Kn\pi r}{2d} \right) dr, \right.$$

A_n, B_n being given by (8).

Numerical Calculation

Let us calculate the stresses on $z = 0$ plane in the crystal topaz due to dislocation.

For this material, we have

$$c'_{44} = 1100 \text{ dynes/cm}^2, \quad c'_{66} = 1350 \text{ dynes/cm}^2$$

So that
$$K = \sqrt{\frac{22}{27}}$$

In particular, we take $r_1 = 1$ cm, $r_2 = 2$ cm and $d = 10$ cm

The variations of $\tau_{\theta z}$ on $z = 0$ plane with the variation of r for different values of l are given in table 1.

Table I

r (in cm)		1.0	1.1	1.5	2.0
$\frac{\pi^3}{8 c'_{44} b} [\tau_{\theta z}]_{z=0}$ (in dynes/cm ^{2})	(*i*) $l = -2$	38 349	34.746	25.242	18.838
	(*ii*) $l = 0$ Homogeneous case	-4 198	-4.447	-6.703	-9.158
	(*iii*) $l = 2$	-25.149	-29.865	-78.908	-188 797

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